Shape Modeling and Geometry Processing

(Discrete) Differential Geometry

Planar Curves – part 2

INTERACTIVE GEOMETRY LAB

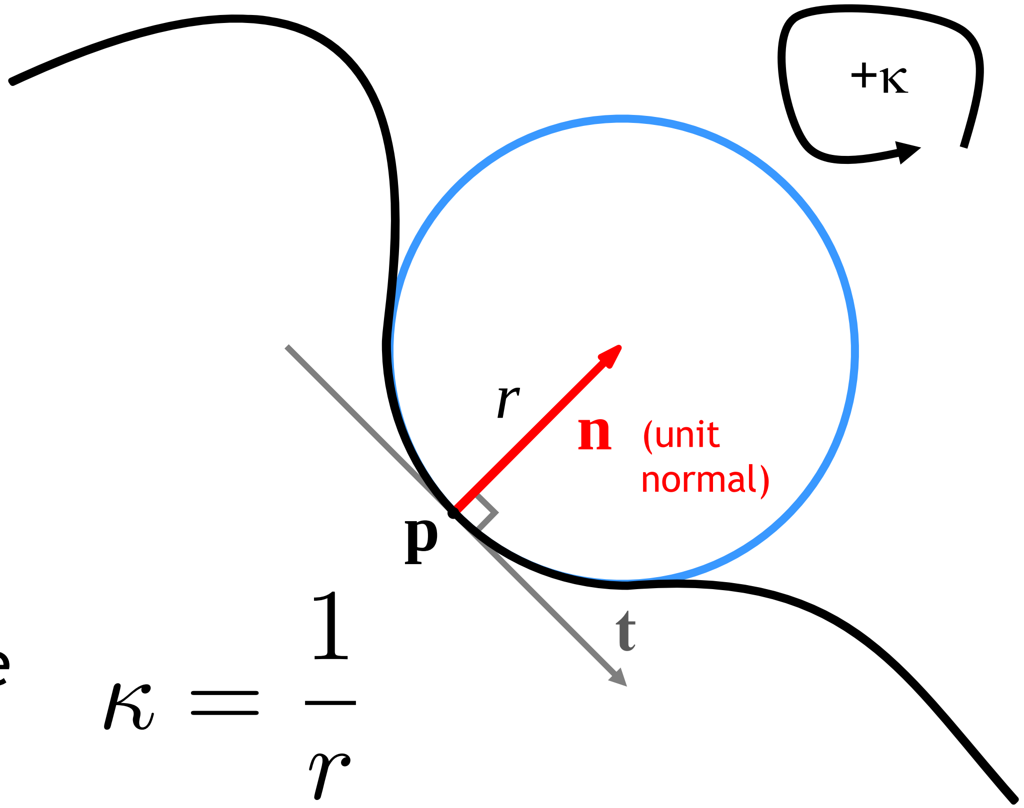
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

RECAP: Curvature in arc-length parameterization

- Curvature κ corresponds to the rate of change of the tangent $\mathbf{t}$ (size of its derivative)

- Curvature is inversely proportional to the osculating circle radius r

$$\kappa = \frac{1}{r}$$



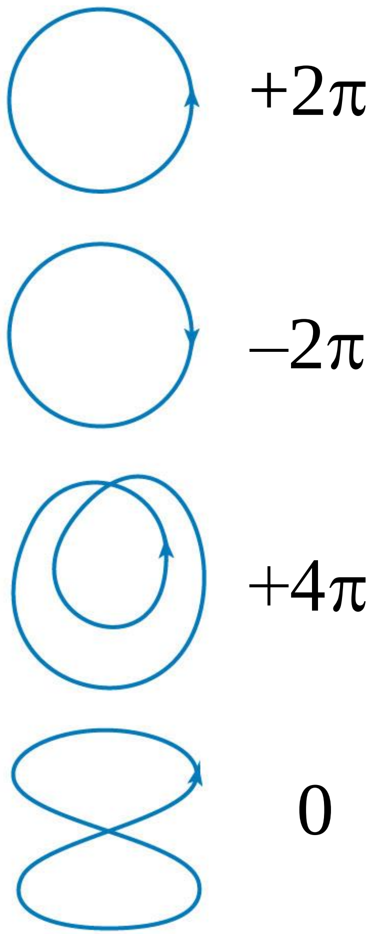
RECAP: Curvature and Topology

Turning Number Theorem:
For a closed curve,
the integral of curvature is
an integer multiple of 2π.

$$\int_\gamma \kappa \, dt = 2\pi k$$

Interpretation: If you want to drive back to the start, your total curvature / steering needs to match the number of loops times 2π.

$$\int_\gamma \kappa \, dt =$$



Recap: Total Curvature

- What is the total curvature of the following curve?

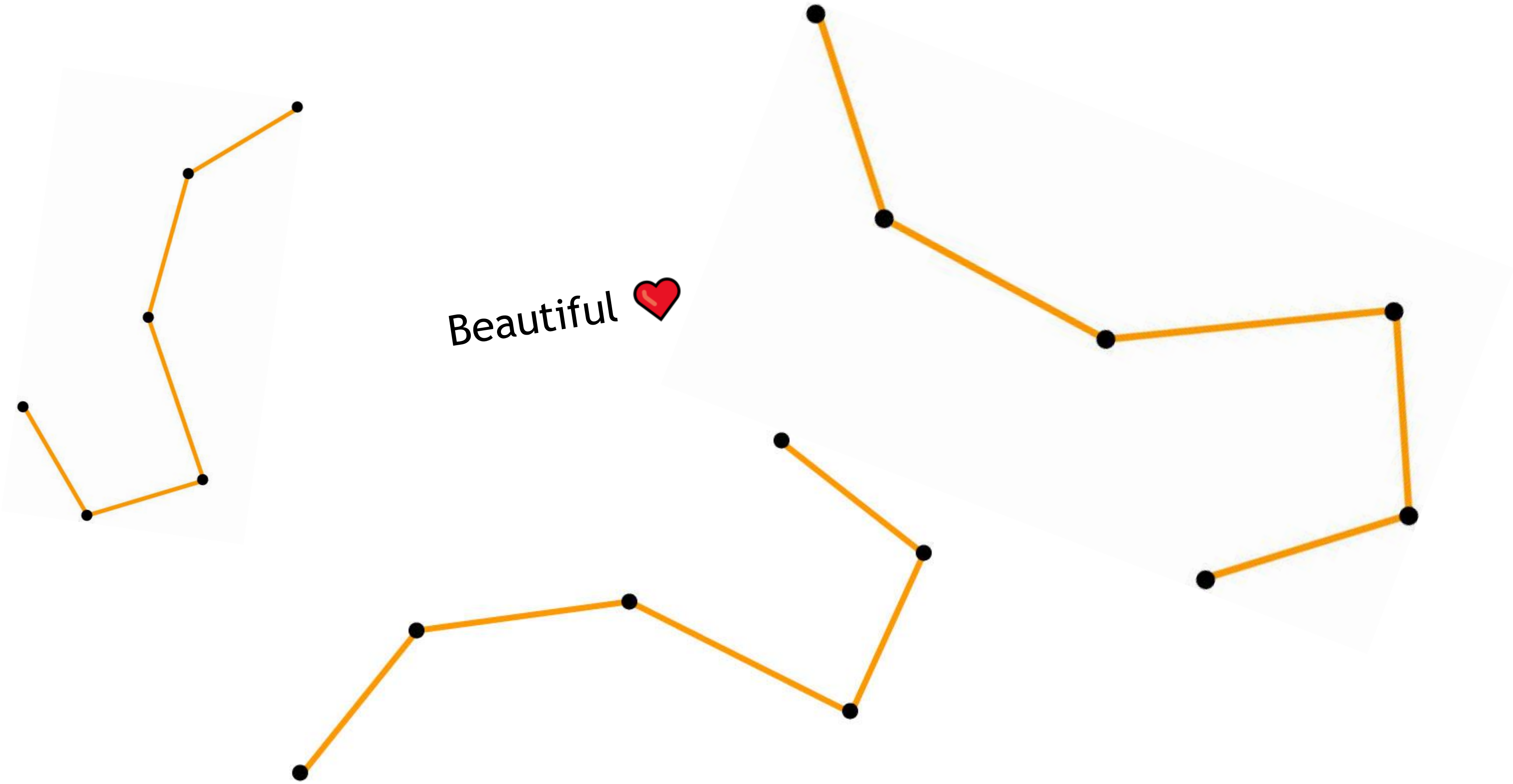
$$\int_\gamma \kappa \, dt = 0$$

Some references: see
http://ddg.cs.columbia.edu/

(Discrete 🤫)

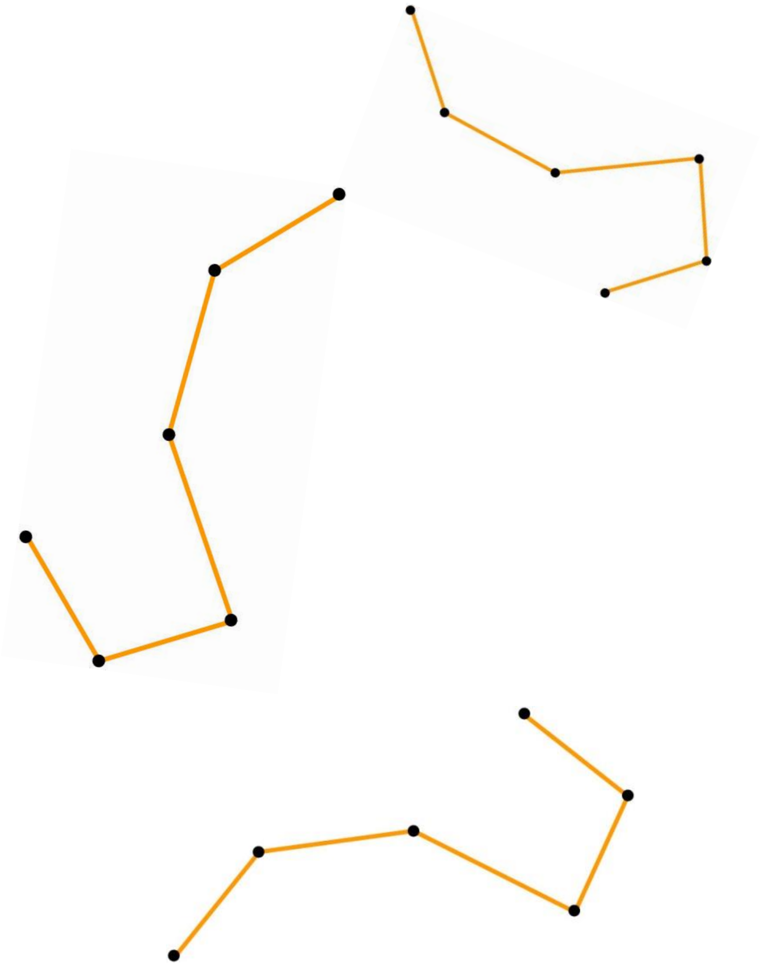
DDG – Curves

Discrete Planar Curves



Discrete Planar Curves

- Piecewise linear curves
- Not smooth at vertices
- Can't take derivatives

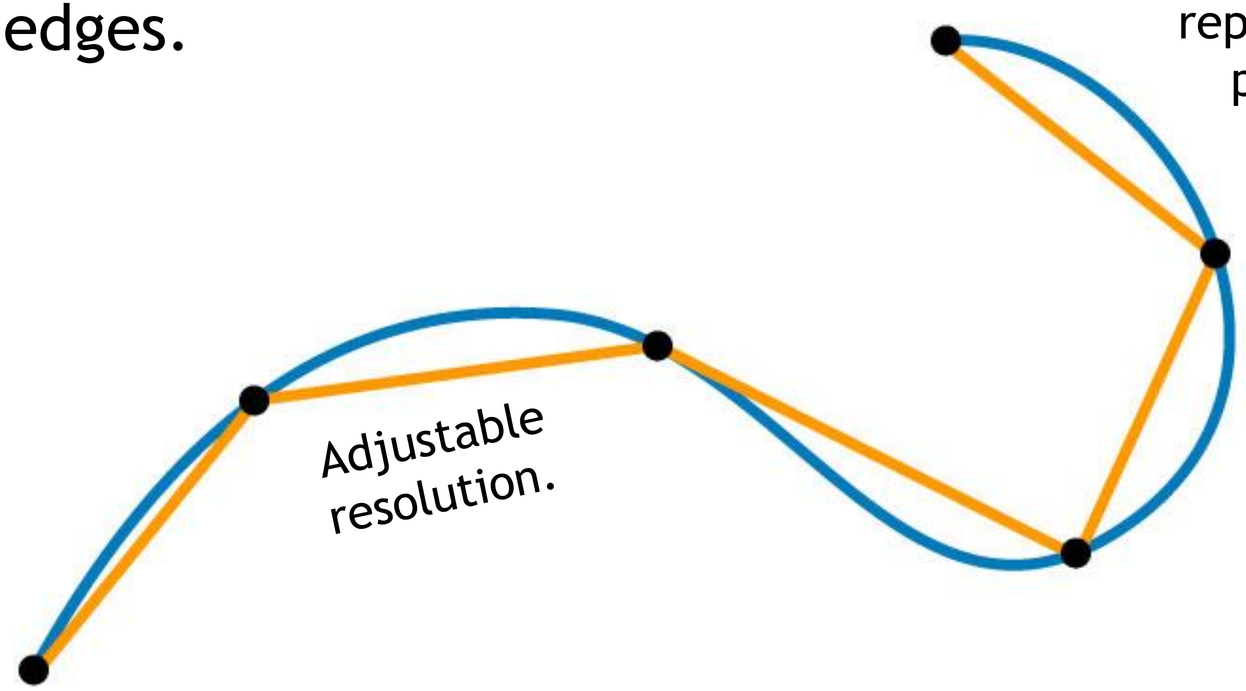


Generalize notions from the smooth world for the discrete case!

Inscribed Polygon, p

- Approximation of the smooth curve.
- Finite number of vertices
  each lying on the curve,
  connected by straight edges.

Many discrete curves approximate the same smooth curve. 😮

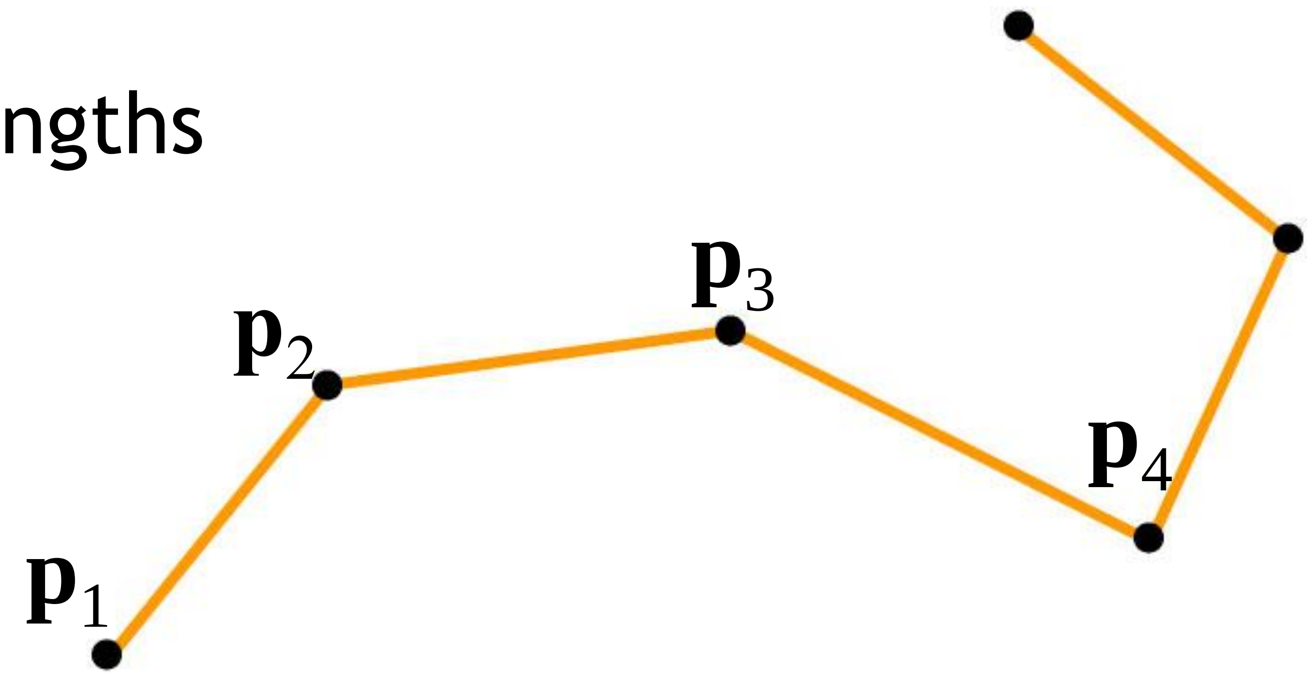


The Length of a Discrete Curve

$$\text{len}(p) = \sum_{i=1}^{n-1} \|\mathbf{p}_{i+1} - \mathbf{p}_i\|$$

- Sum of edge lengths

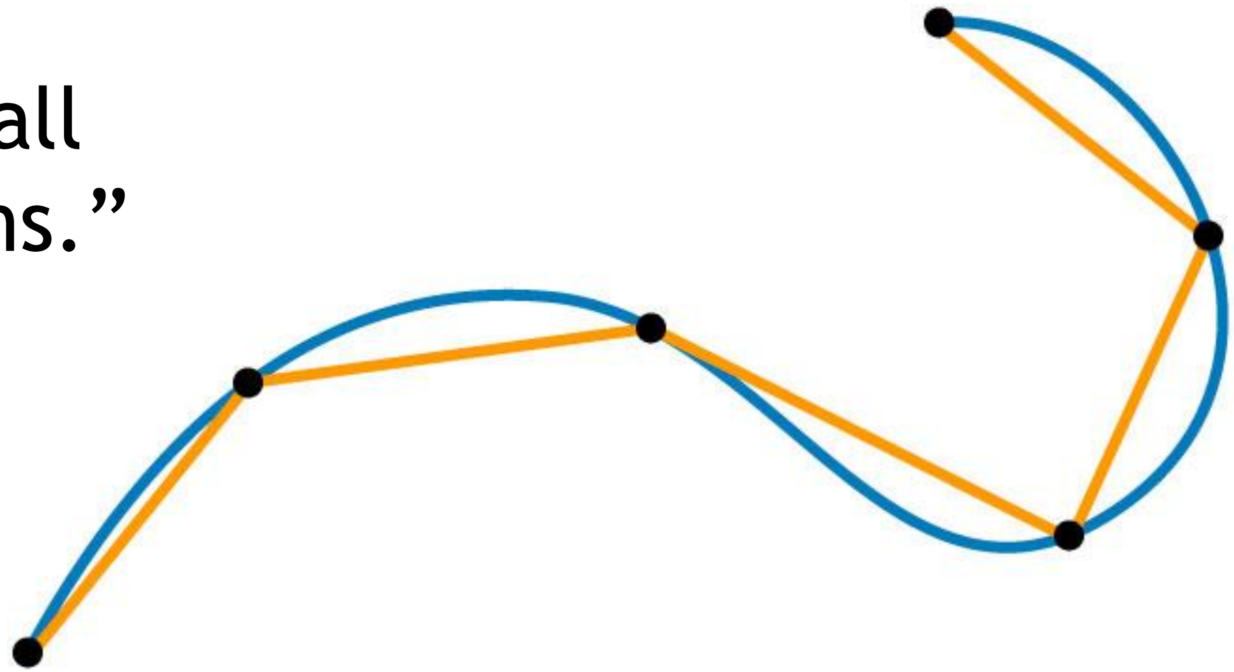
(easy 🙂)



The Length of a Continuous Curve

$$\sup_p \operatorname{len}(p)$$

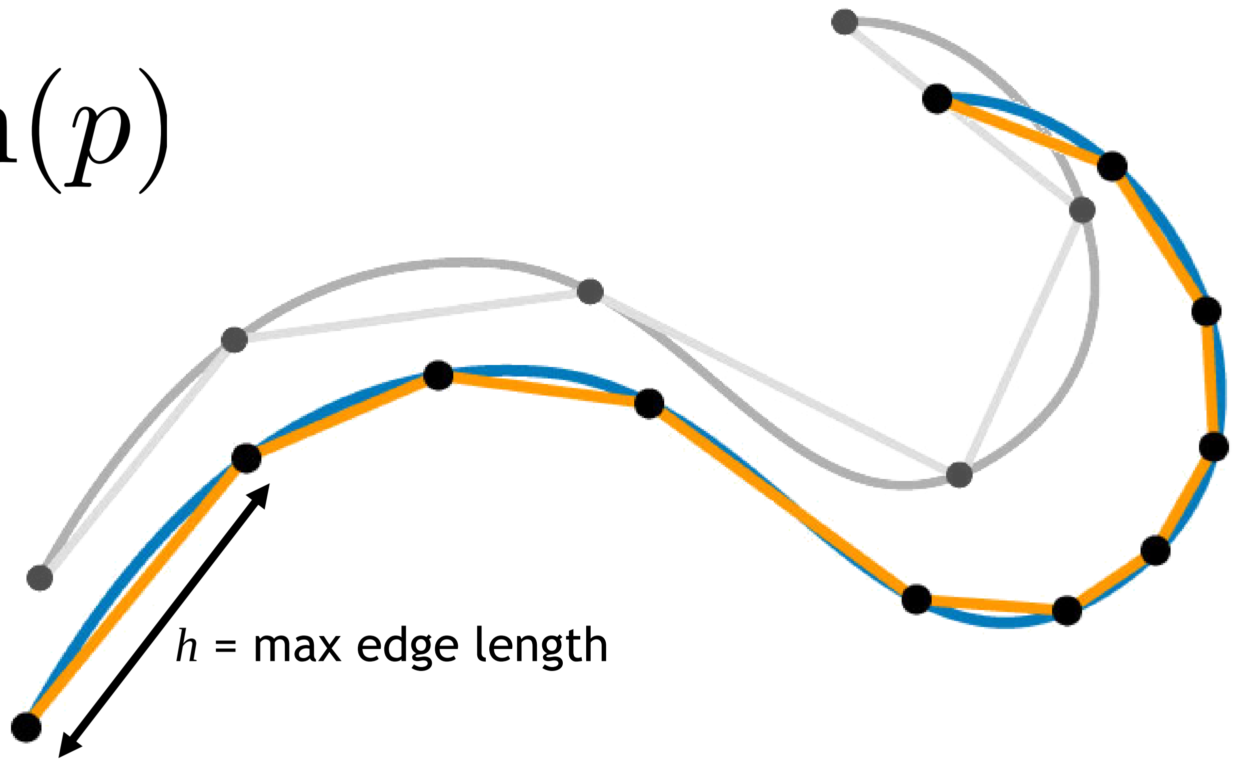
- "Limit length of all inscribed polygons."



The Length of a Continuous Curve

- …take limit over a refinement sequence

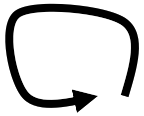
$$\lim_{h \to 0} \operatorname{len}(p)$$

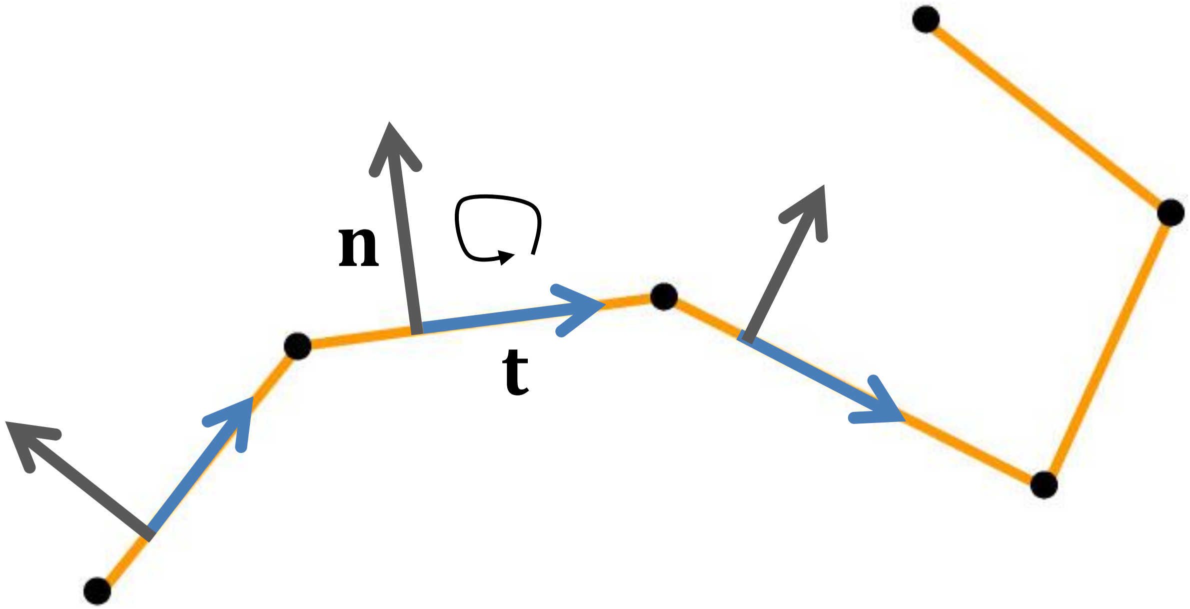


Often any polygon resolution is not enough 😢

(infinite resolution need)

Tangents, Normals

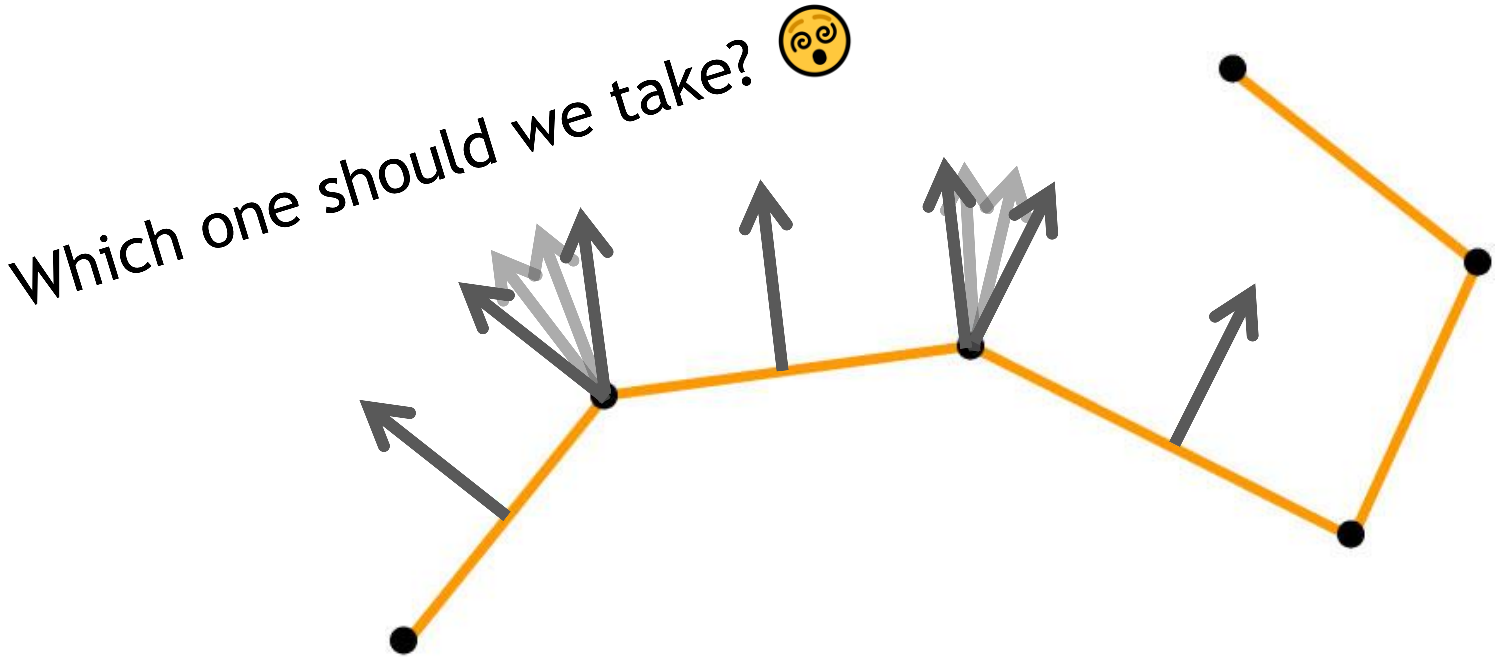
- For any point on the edge,
  - tangent $\mathbf{t}$ = unit vector along the 
  - normal $\mathbf{n}$ = tangent vector rotated by 90° anti-cloclwise



Tangents, Normals

- For vertices, we have many options

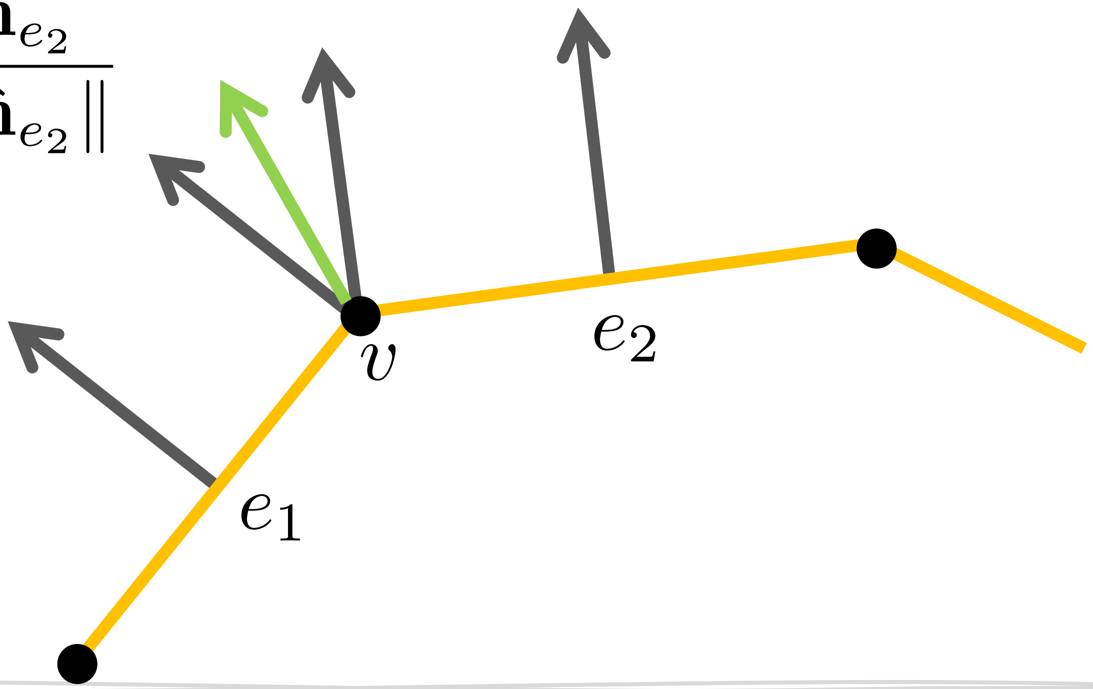
There is no “obvious” choice!



Tangents, Normals

- Can choose to average the adjacent edge normals

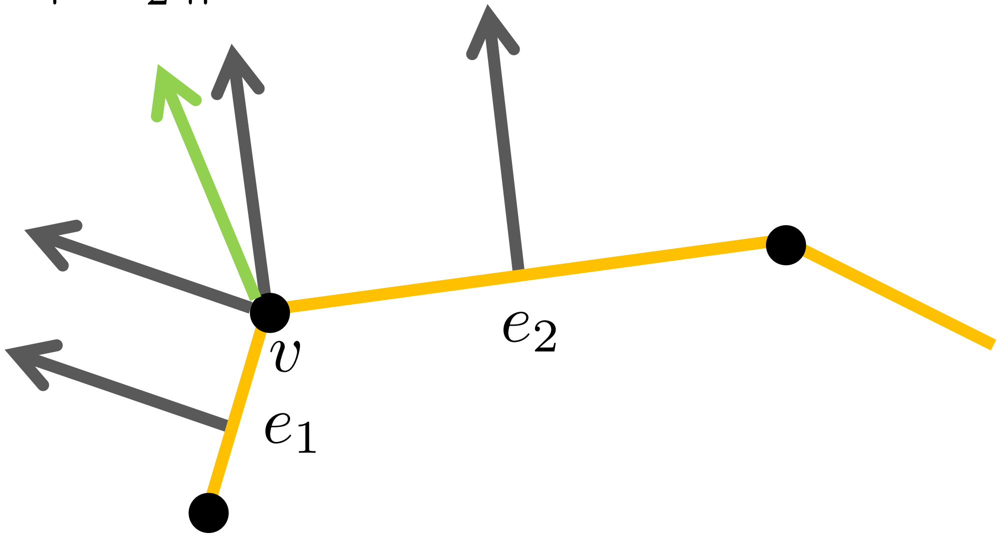
$$\hat{\mathbf{n}}_v = \frac{\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}}{\|\hat{\mathbf{n}}_{e_1} + \hat{\mathbf{n}}_{e_2}\|}$$



Tangents, Normals

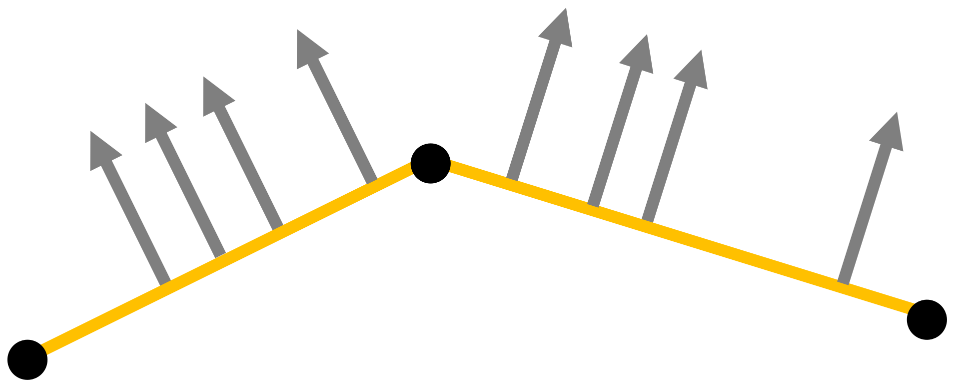
- Weighting by edge lengths

$$\hat{\mathbf{n}}_v = \frac{|e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}}{\||e_1|\hat{\mathbf{n}}_{e_1} + |e_2|\hat{\mathbf{n}}_{e_2}\|}$$



Curvature of a Discrete Curve

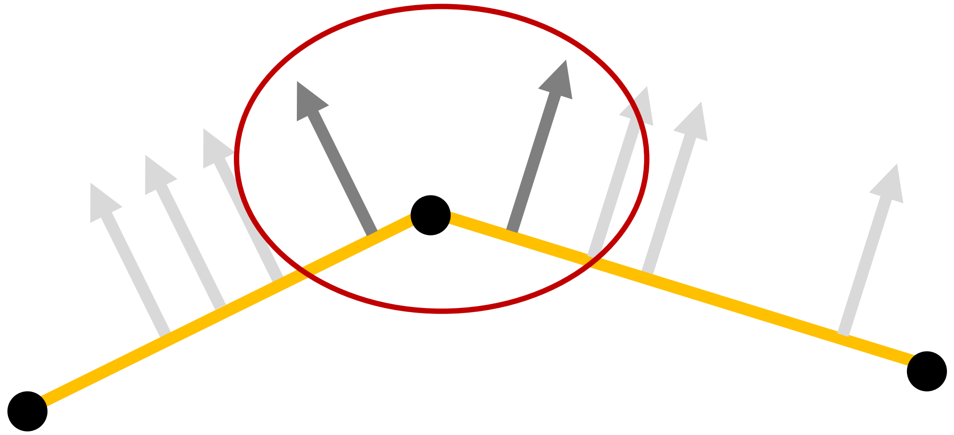
- Curvature is the amount of change in normal direction as we travel along the curve



no change along each edge –
curvature is zero along edges 😴

Curvature of a Discrete Curve

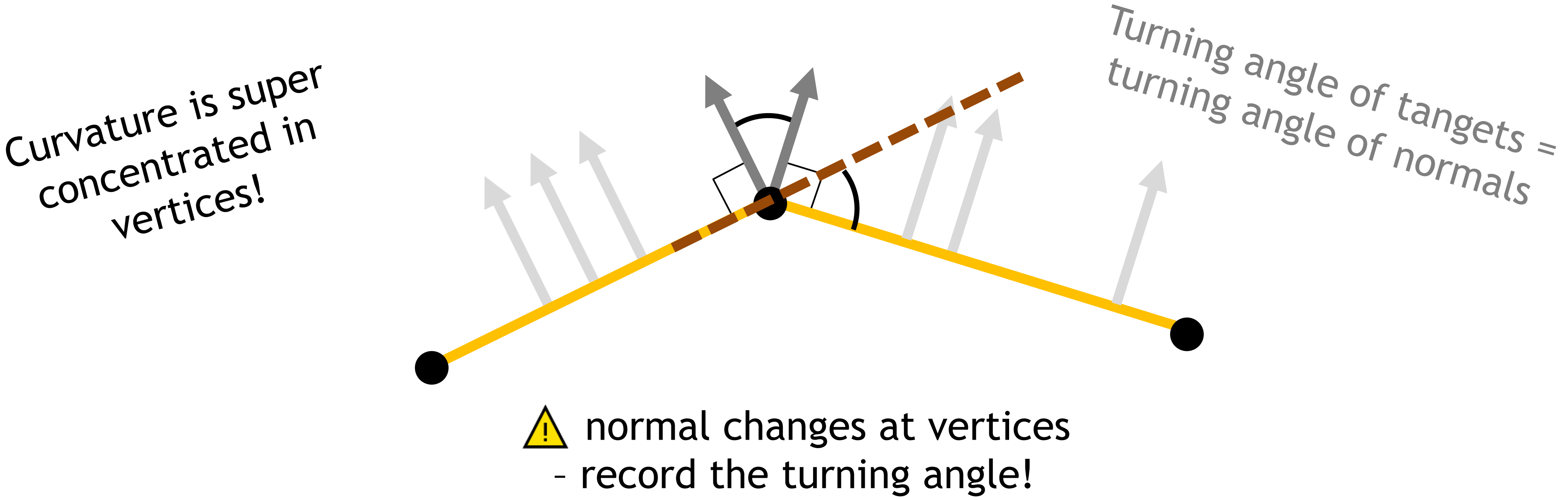
- Curvature is the amount of change in normal direction as we travel along the curve



⚠ normal changes at vertices
– record the turning angle!

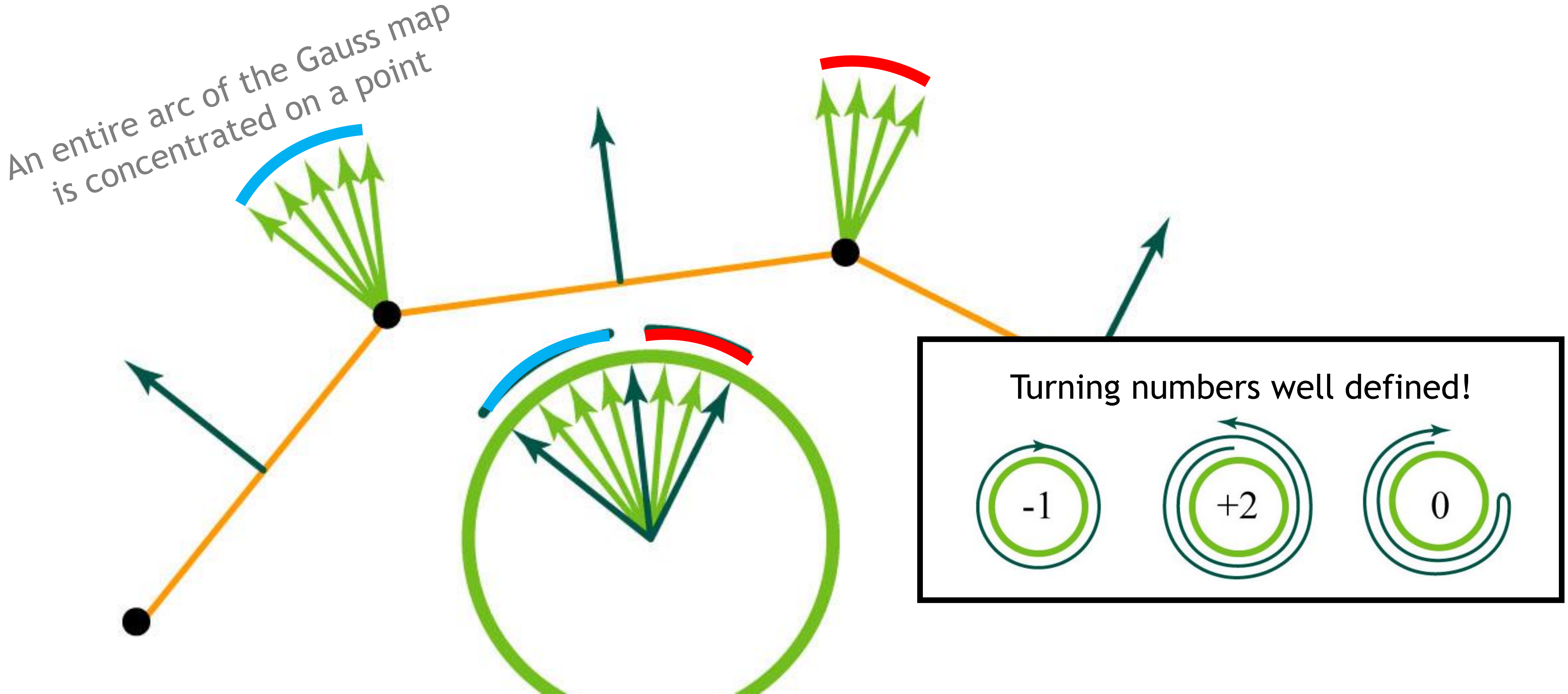
Curvature of a Discrete Curve

- Curvature is the amount of change in normal direction as we travel along the curve



Discrete Gauss Map

- Edges map to points, vertices map to arcs.



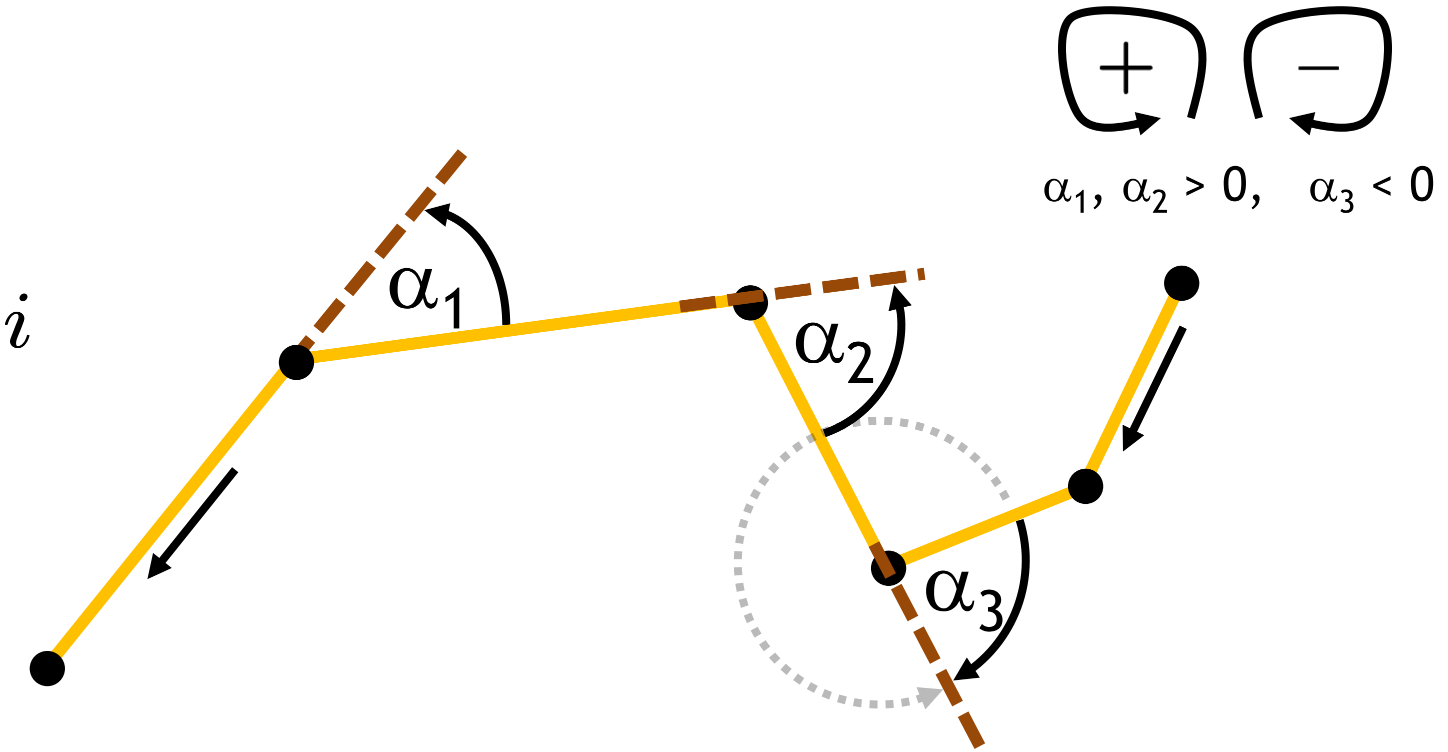
Curvature of a Discrete Curve

- Gauss Map and turning angle constant along the edges
- Turning angle at the vertices = the change in normal direction

$$\mathrm{tsc}(p) = \sum_{i=1}^{n} \alpha_i$$

Total signed curvature

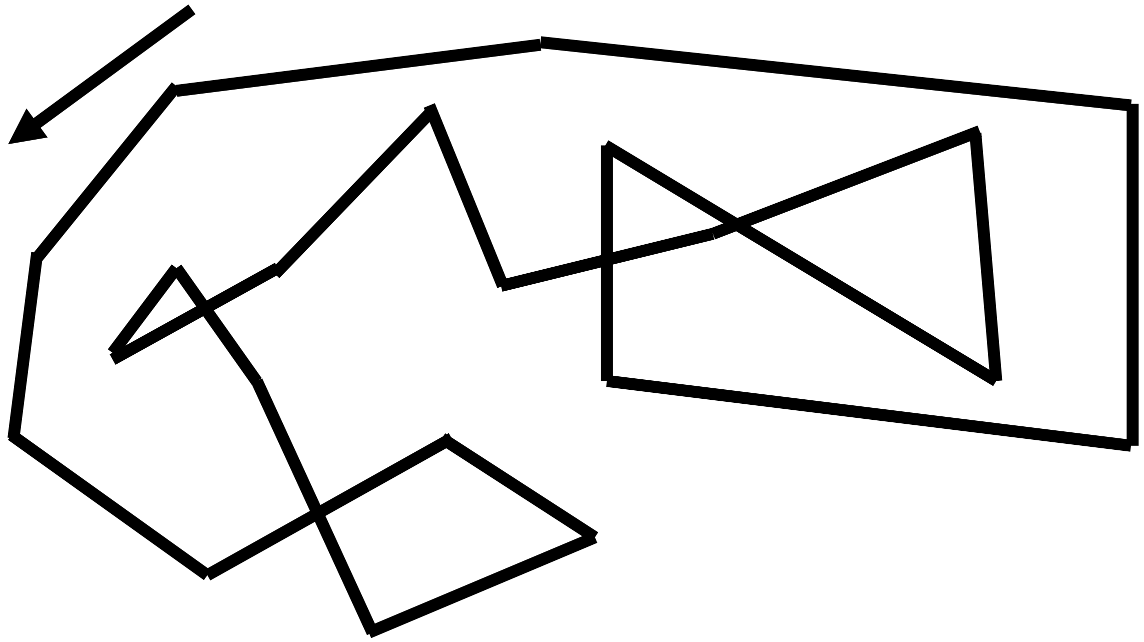
Sum of turning angles



Total Curvature

- What is the total signed curvature of the following curve?

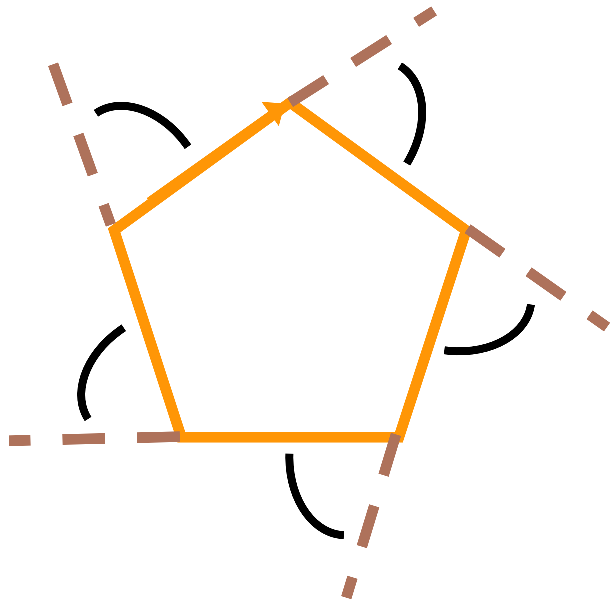
$$\mathrm{tsc}(p) = \sum_{i=1}^{n} \alpha_i \ = ?$$



Discrete Turning Number Theorem

Discrete Turning Number Theorem:

$$\mathrm{tsc}(p) = \sum_{i=1}^{n} \alpha_i = \ ?$$

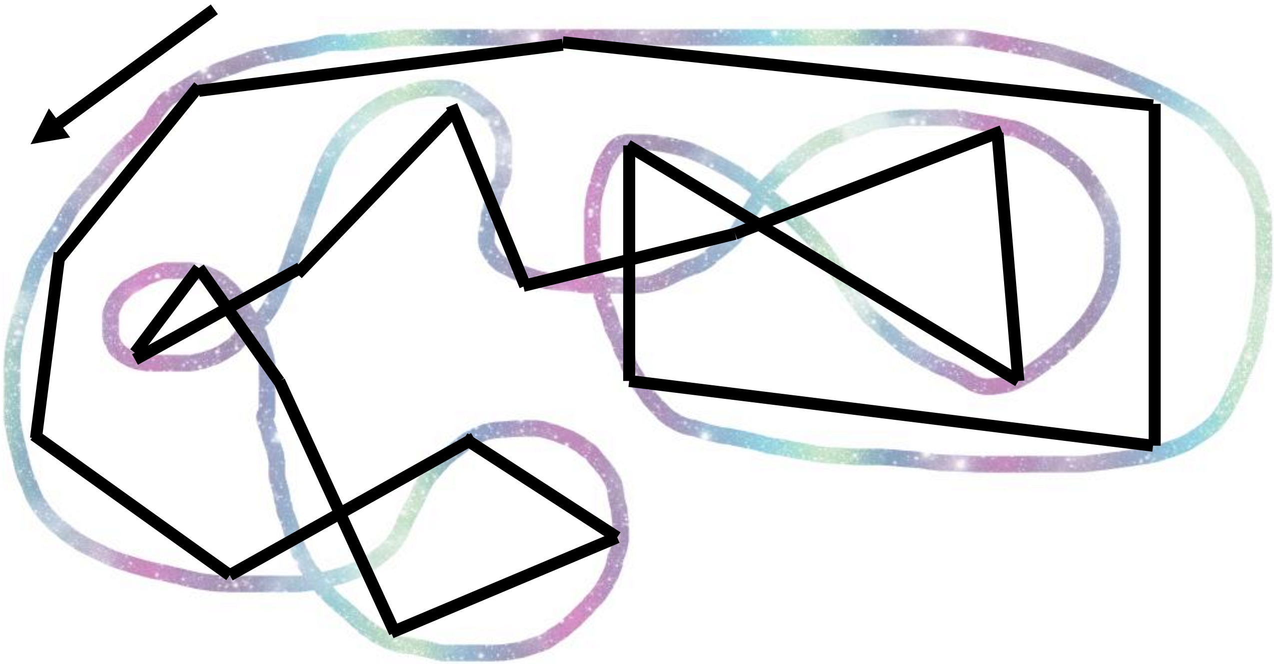


Total Curvature

- What is the total curvature of the following curve?

$$\text{tsc}(p) = \sum_{i=1}^{n} \alpha_i = 0$$

$$\int_\gamma \kappa \, dt = 0$$



Turning Number Theorem

Continuous world

$$\int_\gamma \kappa \, dt = 2\pi k$$

Discrete world

$$\sum_{i=1}^{n} \alpha_i = 2\pi k$$

k:

$$\kappa = \alpha_i \quad ??$$

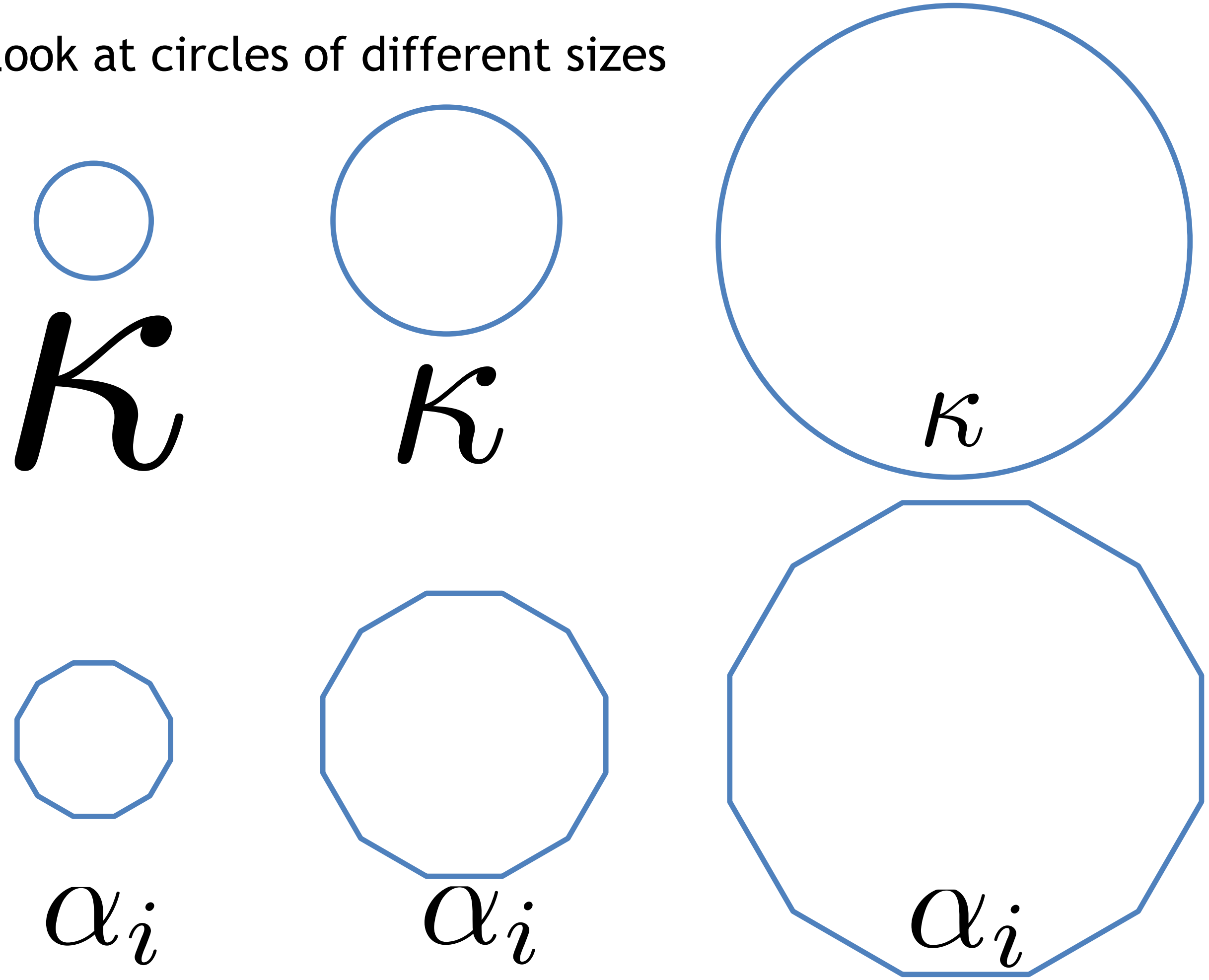
Is this a good choice of discretization?

Curvature is scale dependent

Let's look at circles of different sizes

There is a problem...

$$\kappa = \frac{1}{r}$$

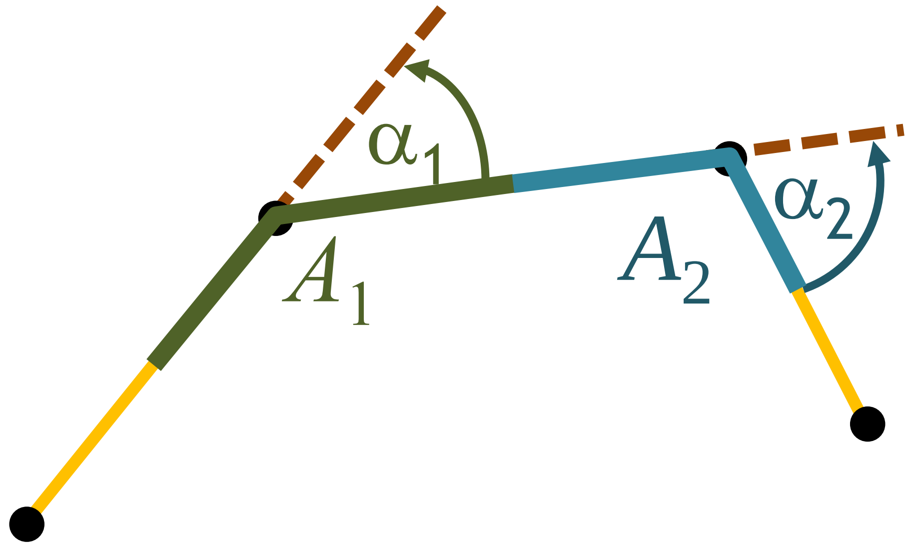


- ✓ κ is scale-dependent
- ✗ α_i is scale-independent

Discrete Curvature – Integrated Quantity!

- We cannot view α_i as pointwise curvature
- It is ***integrated curvature*** over a local area associated with vertex i

$$\alpha_1 = A_1 \cdot \kappa_1$$
$$\alpha_2 = A_2 \cdot \kappa_2$$
$$\sum A_i = \mathrm{len}(p)$$



The vertex areas A_i form a covering of the curve. They are pairwise disjoint (except endpoints).

Turning Number Theorem

Continuous world

$$\int_\gamma \kappa\, dt = 2\pi k$$

Discrete world

$$\sum_{i=1}^{n} \alpha_i = 2\pi k$$

k:

$$\kappa\, dt = \alpha_i$$

Length elements

Much better!

Structure Preservation

- For arbitrary discrete curves:
  - Total signed curvature obeys discrete turning number theorem

    even coarse curves

Preservation: ↔ discrete analogue of continuous theorem

Which other continuous theorems to preserve?

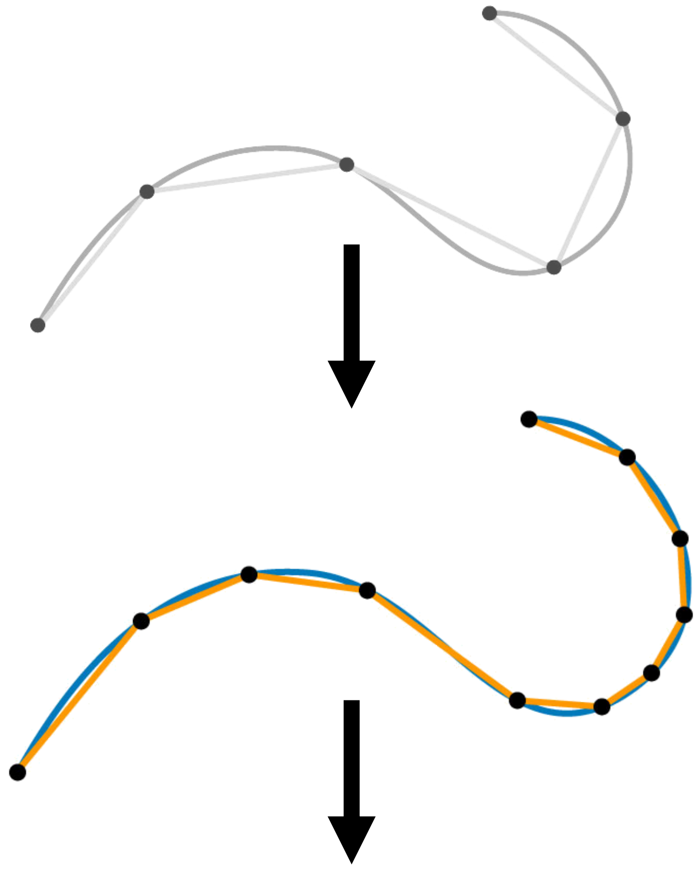
That depends on the application…

Convergence

- Consider a curve refinement sequence

$$\lim_{\text{refinement}} length(\, polygon\,) = length(\, smooth\ curve)$$

Ideally:
discrete measures approaches continuous analogue when refining.

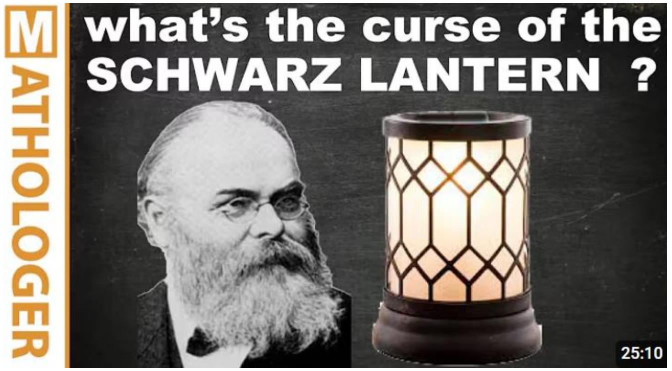


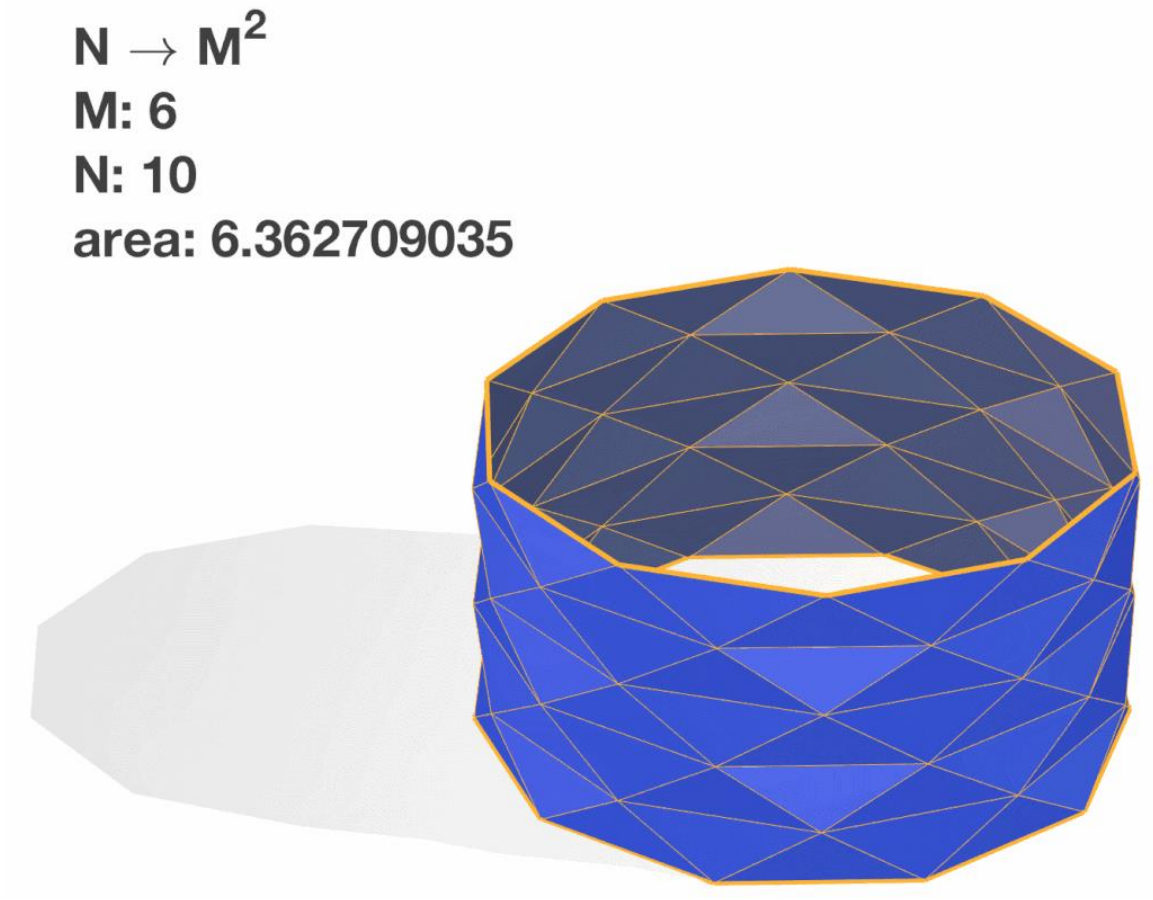
Questions:

- Which refinement sequence?
  - depends on discrete operator
  - pathological sequences may exist (e.g. Schwarz latern)
- In what sense does the operator converge?
  - pointwise, L_2, linear, quadratic

Pathological Sequences Example

A mesh that converges to the cylinder but has a different area.



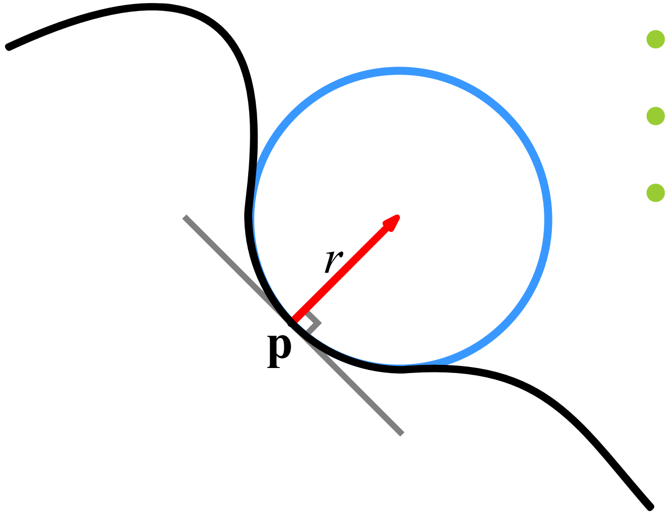


Schwarz lantern area convergence (or lack thereof) for different refinement strategies. (Wikimedia)

Another option for curvature

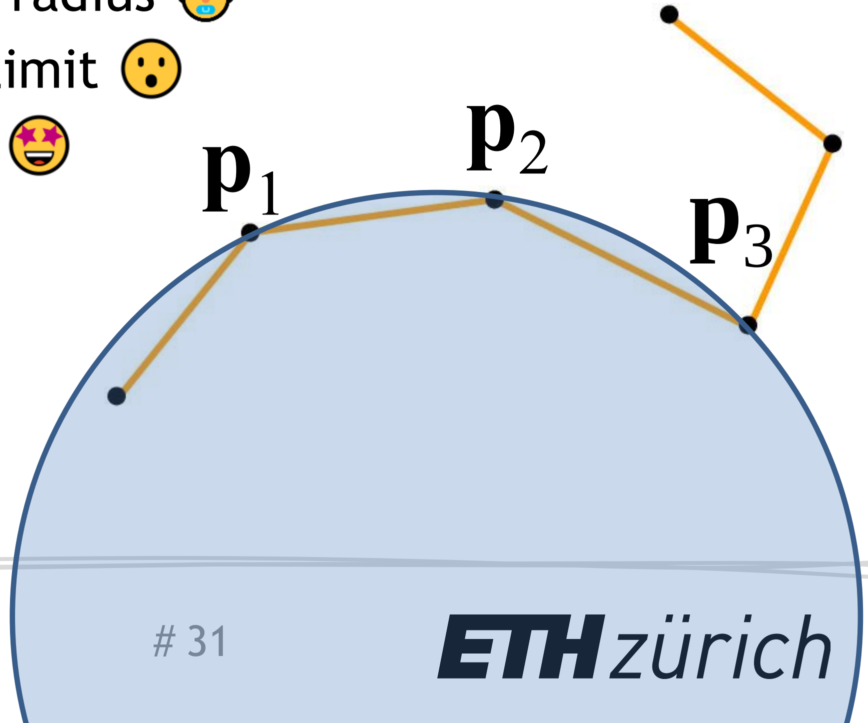
- Alternative discrete curvature based on oscillating circle relation.

$$\kappa = \frac{1}{r}$$



- Pass a circle through 3 points, take 1/radius 👶
- Equal to the angle in the refinement limit 😮
- Better accuracy (faster convergence) 🤩
- But no turning number theorem 😓

Still, in practice this is often the most convenient discrete curvature definition. 👍



Recap

Convergence based approach:

vs.

Structure-preserving approach:

Converging to the smooth equivalent when refining the mesh.

Generally easier but sometimes with approximation issues

Traditional numerical analysis 😴

EXACT property preservations even on coarse meshes

e.g. discrete turning number theorem (Generally harder to achieve)

Discrete Differential Geometry (DDG) 😎

Shape Modeling and Geometry Processing

(Discrete) Differential Geometry Surfaces

INTERACTIVE GEOMETRY LAB

March 19, 2025

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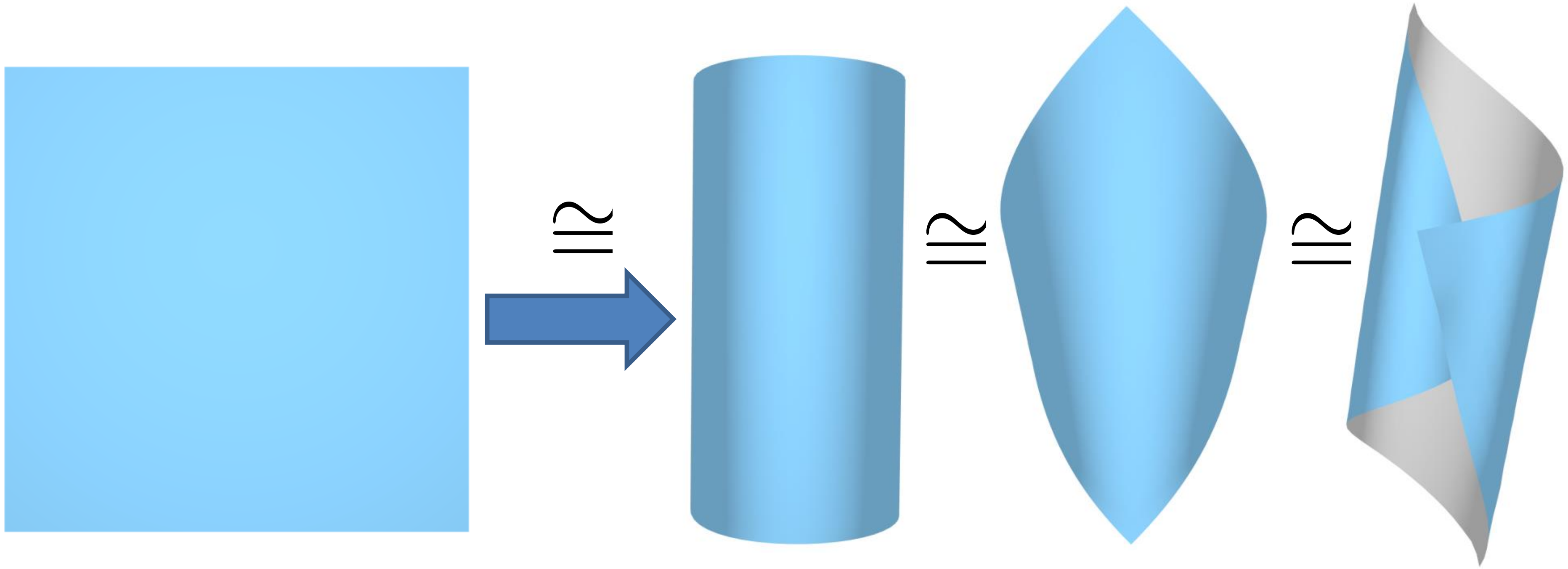
Intrinsic and extrinsic properties

Planar sheet

Isometric shapes



Shaped without any distortion

Intrinsic and extrinsic properties

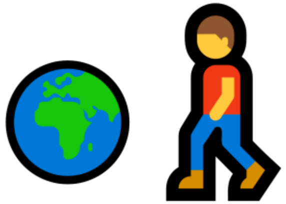
- Intrinsic properties: preserved under isometry
  - Distance on surface
  - Angles on surface

What a tiny tiny Ant living on the surface would notice.

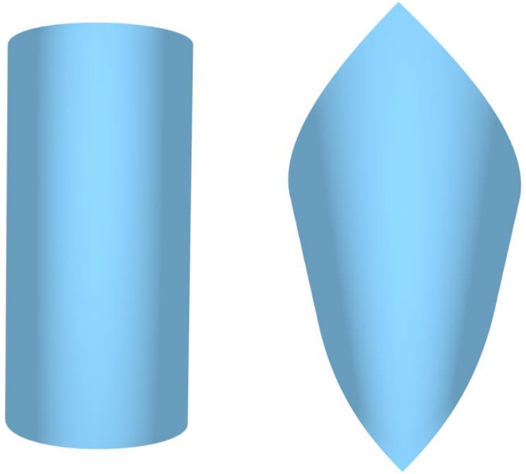
- Extrinsic properties: depend on the embedding
  - Tangents
  - Normals
  - Curvature

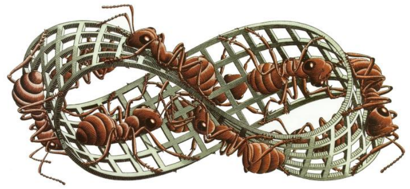
What is curvature on a surface?

What a big outside observer would notice.



Isometric
=
intrinsically the same





Surfaces, Parametric Form

- Continuous surface

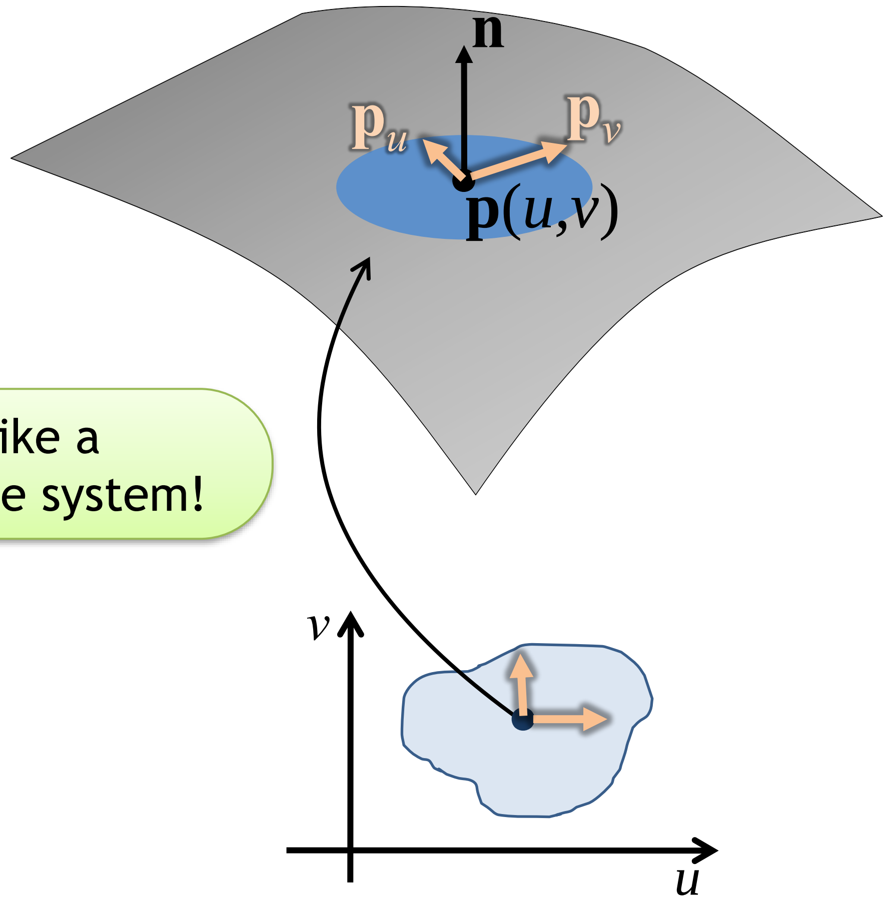
$$\mathbf{p}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in \mathbb{R}^2$$

- Tangent plane at point $\mathbf{p}(u,v)$ is spanned by

$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \quad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

⚠ These vectors don't have to be orthogonal

It's like a coordinate system!



Isoparametric Lines

- Lines mapped on the surface when keeping one parameter fixed in the parametrization

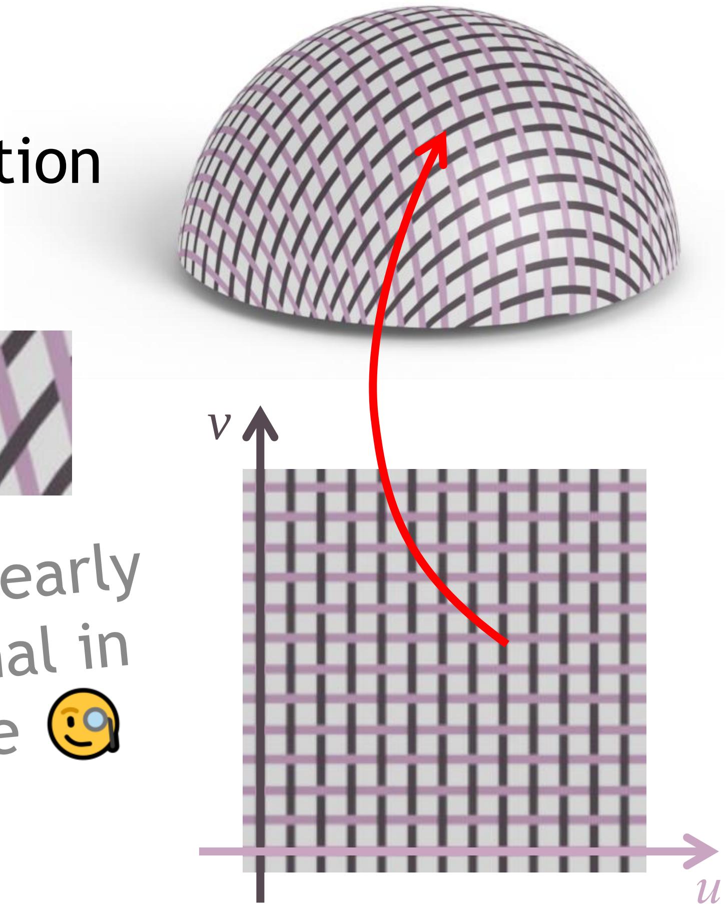
$$\gamma_{u_0}(v) = \mathbf{p}(u_0, v)$$

u fixed

$$\gamma_{v_0}(u) = \mathbf{p}(u, v_0)$$

v fixed

Derivatives clearly not orthogonal in this example 🧐



Intrinsic Geometry

- **First *fundamental* form**

Important! 😮

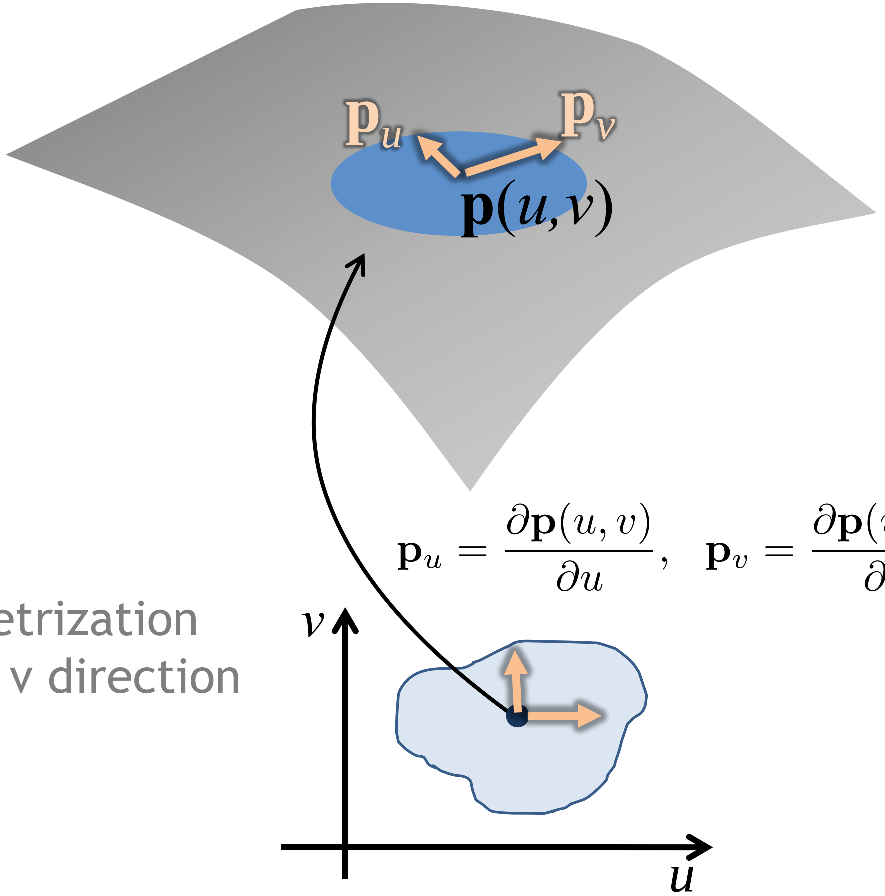
Parametrization speed of u direction

Alignment of parametrizations

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{p}_u^\top \mathbf{p}_u & \mathbf{p}_u^\top \mathbf{p}_v \\ \mathbf{p}_u^\top \mathbf{p}_v & \mathbf{p}_v^\top \mathbf{p}_v \end{pmatrix}$$

Alignment of parametrizations

Parametrization speed of v direction



$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u,v)}{\partial u}, \quad \mathbf{p}_v = \frac{\partial \mathbf{p}(u,v)}{\partial v}$$

- Needed to define lengths, angles and areas.
- It defines the *metric* of the surface.

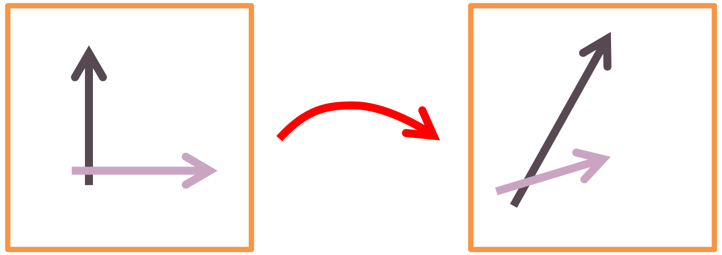
The First Fundamental Form

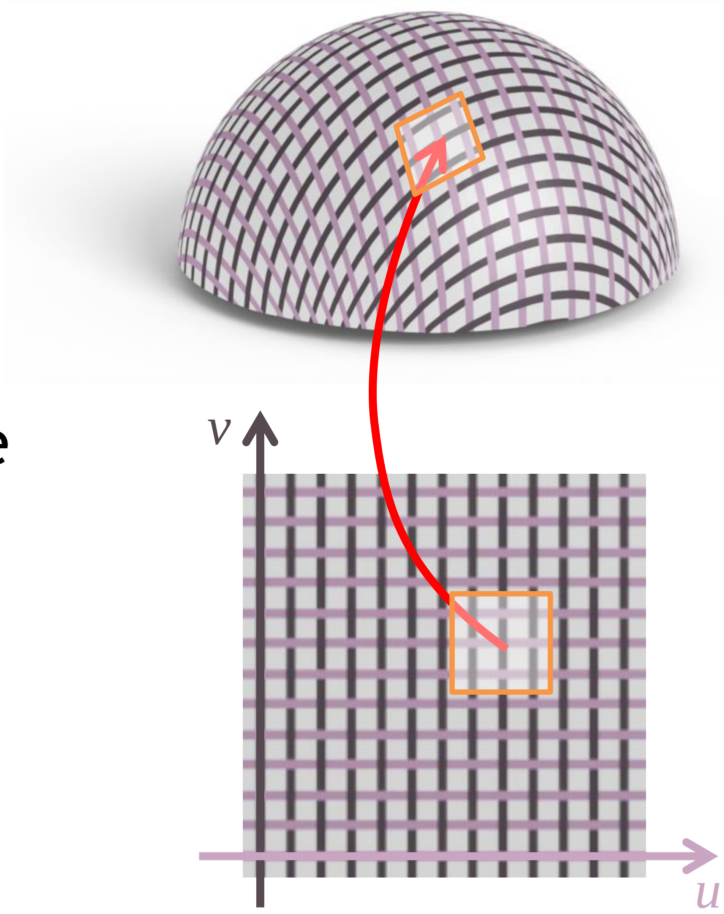
- First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{p}_u^\mathsf{T}\mathbf{p}_u & \mathbf{p}_u^\mathsf{T}\mathbf{p}_v \\ \mathbf{p}_u^\mathsf{T}\mathbf{p}_v & \mathbf{p}_v^\mathsf{T}\mathbf{p}_v \end{pmatrix}$$

- Maps the canonical uv-plane to the tangent plane

- Defines a scalar product in the uv-plane





The First Fundamental Form

- $\mathbf{I}$ allows to measure
  - length, angles, area on the surface ❤️
  - arc element

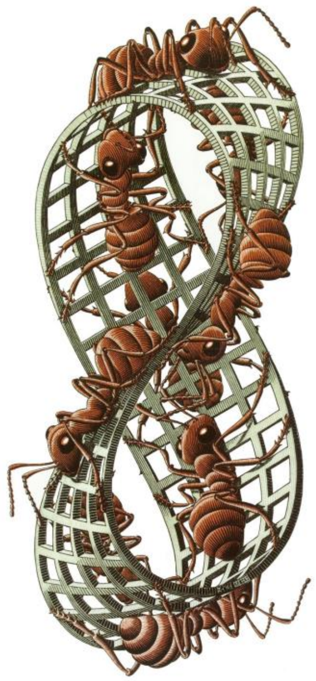
$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2$$

  - area element

$$dA = \sqrt{EG - F^2}\,dudv$$

determinant of $\mathbf{I}$

  - Anything the tiny ant can measure 🐜



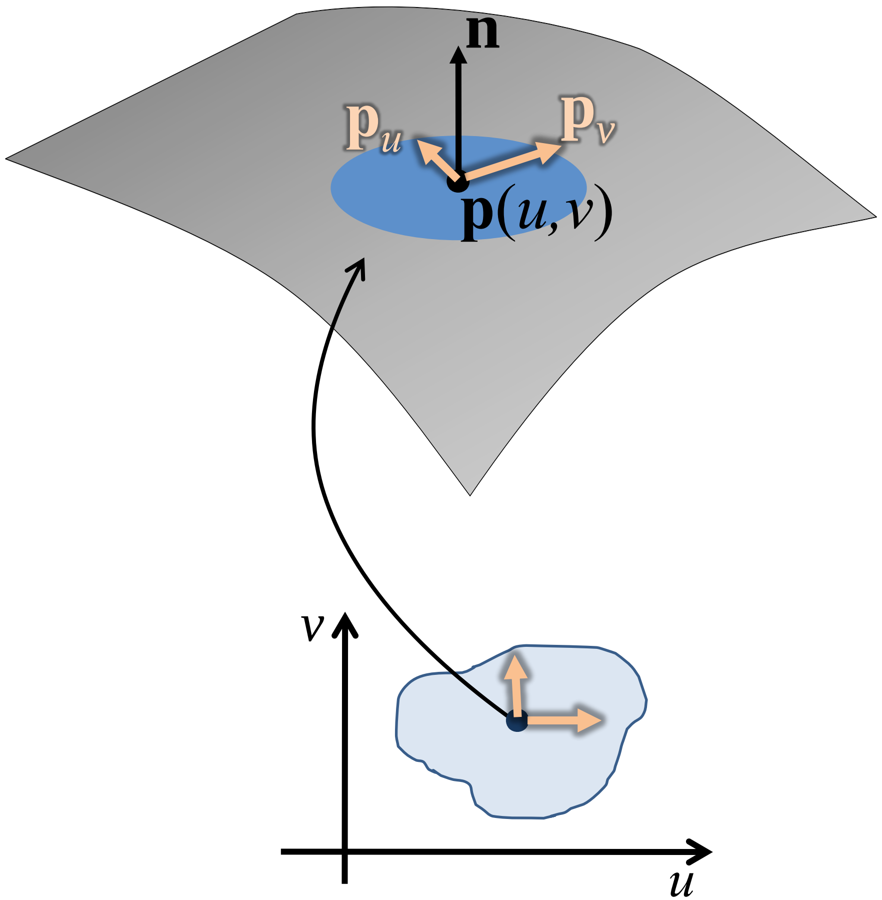
Surface Normals

- Surface normal:

$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

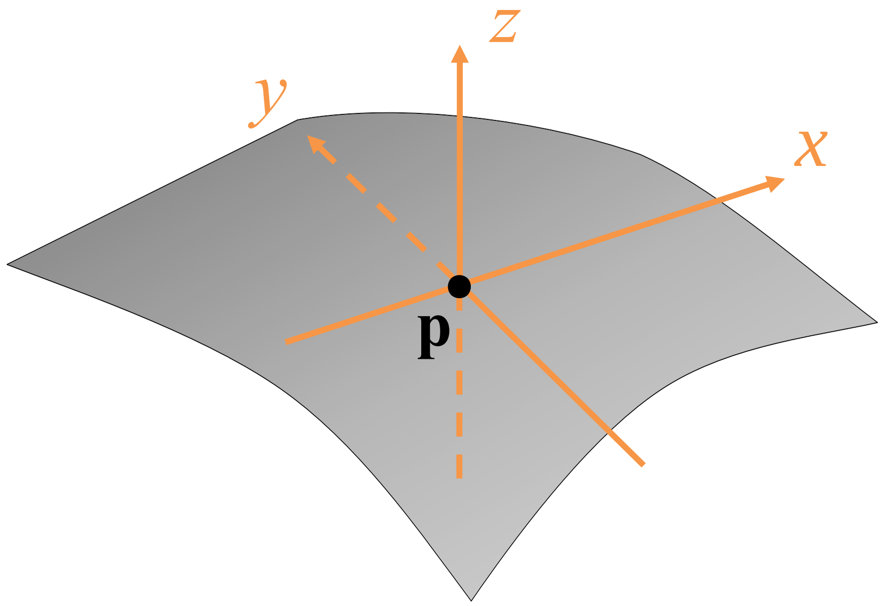
⚠ Assuming *regular* parameterization.

Definition: $\mathbf{p}_u \times \mathbf{p}_v \neq 0$



The Second Fundamental Form

- Local coordinate frame *xyz*:
  tangets $\mathbf{p}_u$, $\mathbf{p}_v$ and normal $\mathbf{n}$
- The surface is locally a **height field** w.r.t. the tangent plane $z = z(x,y)$
- The height field can be locally approximated by a quadric:



$$z \approx 0.5\, e\, x^2 + f\, xy + 0.5\, g\, y^2$$

„think 2nd order Taylor expansion“

$$\mathrm{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{uu}^\top \mathbf{n} & \mathbf{p}_{uv}^\top \mathbf{n} \\ \mathbf{p}_{uv}^\top \mathbf{n} & \mathbf{p}_{vv}^\top \mathbf{n} \end{pmatrix}$$

The Second Fundamental Form

Out of plane bending in *u* direction.

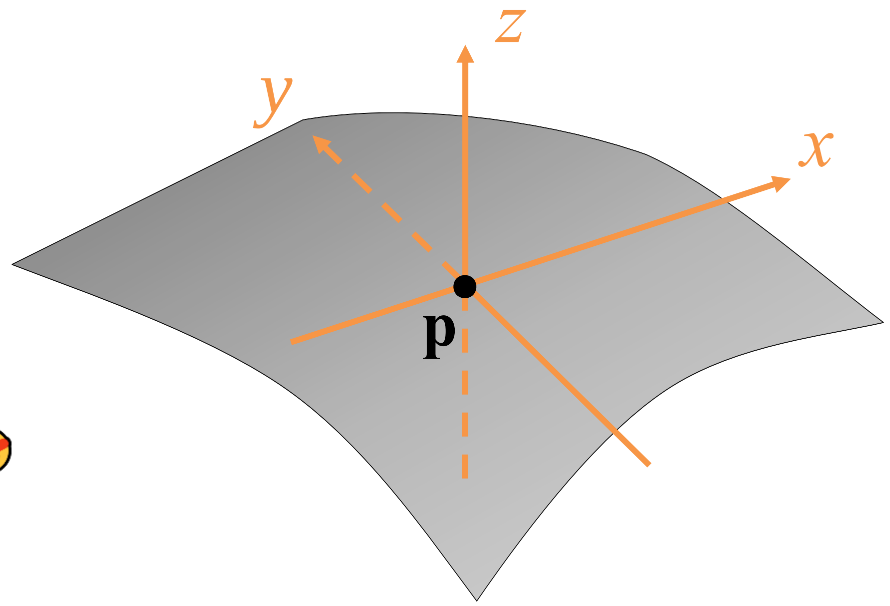
Mixed curvature interaction

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{uu}^\top \mathbf{n} & \mathbf{p}_{uv}^\top \mathbf{n} \\ \mathbf{p}_{uv}^\top \mathbf{n} & \mathbf{p}_{vv}^\top \mathbf{n} \end{pmatrix}$$

Beautiful! 😍

Mixed curvature interaction

Out of plane bending in *v* direction.



„How is my surface bend relative to the normal plane?"

Fundamental Forms

- First fundamental form (first derivative surface behavior)

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{p}_u^\mathsf{T}\mathbf{p}_u & \mathbf{p}_u^\mathsf{T}\mathbf{p}_v \\ \mathbf{p}_u^\mathsf{T}\mathbf{p}_v & \mathbf{p}_v^\mathsf{T}\mathbf{p}_v \end{pmatrix}$$

- Second fundamental form (second derivative surface behavior)

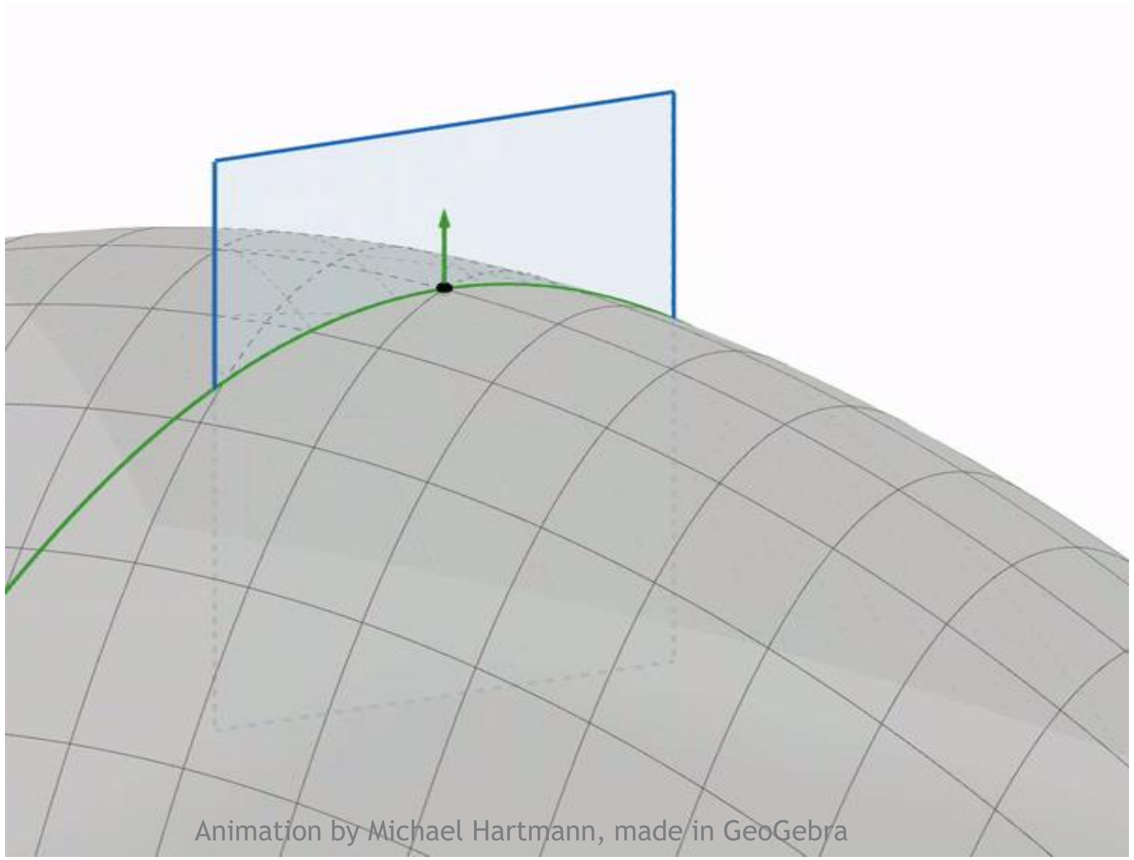
$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{uu}^\mathsf{T}\mathbf{n} & \mathbf{p}_{uv}^\mathsf{T}\mathbf{n} \\ \mathbf{p}_{uv}^\mathsf{T}\mathbf{n} & \mathbf{p}_{vv}^\mathsf{T}\mathbf{n} \end{pmatrix}$$

Together, they **define** a surface!
(if certain compatibility conditions hold,
called Gauss-Codazzi-Mainardi equations).

Compare with curvature of planar curves:
(Curvature determines the entire curve shape)

Directional Normal Curvature

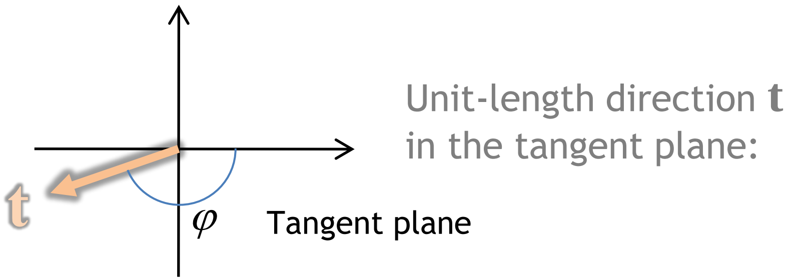
$$\mathbf{n}(u,v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$



Let γ be the intersection curve of the surface with the plane through $\mathbf{n}$ and $\mathbf{t}$.

Normal curvature:

$$\kappa_n(\varphi) = \kappa(\gamma(\mathbf{p}))$$



Directional normal curvature of surface = curvature of the intersection curve passing in this direction

Surface Curvatures

- Principal curvatures
  - Minimal curvature $\kappa_1 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$
  - Maximal curvature $\kappa_2 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$

Reminder:

$\mathbf{II}$ is local *quadratic* approximation of the surface as height field over the tangent plane

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{uu}^{\mathsf{T}}\mathbf{n} & \mathbf{p}_{uv}^{\mathsf{T}}\mathbf{n} \\ \mathbf{p}_{uv}^{\mathsf{T}}\mathbf{n} & \mathbf{p}_{vv}^{\mathsf{T}}\mathbf{n} \end{pmatrix}$$

Surface Curvatures

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{uu}^{\mathsf{T}}\mathbf{n} & \mathbf{p}_{uv}^{\mathsf{T}}\mathbf{n} \\ \mathbf{p}_{uv}^{\mathsf{T}}\mathbf{n} & \mathbf{p}_{vv}^{\mathsf{T}}\mathbf{n} \end{pmatrix}$$

$\boldsymbol{w}^{\mathrm{T}}\mathbf{II}\boldsymbol{w}$ = normal bending in directon $\boldsymbol{w}$, $||\boldsymbol{w}|| = \mathbf{1}$

Theorem:

$\kappa_1 = \kappa_{min}, \kappa_2 = \kappa_{max}$ are eigenvalues of the $\mathbf{II}$

$\mathbf{II}$ is a symmetric matrix – has real eigenvalues and **orthogonal** eigenvectors

Max and min bending directions are always orthogonal! 🤩

(Because symmetric matrices have orthagonal eigenvector basis.)

Thank you